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INVARIANT CONNECTIONS AND YANG-MILLS SOLUTIONS

BY

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ABSTRACT. A condition on the self-duality and the stability of Yang-Mills solutions are discussed. The canonical invariant G-connections on S^4 and $P_2(\mathbf{C})$ are considered as Yang-Mills solutions. The non-self-duality of the connections requires the injectivity of the isotropy homomorphisms. We construct examples of non-self-dual connections on G-vector bundles (G is a compact simple group). Under a certain property of the isotropy homomorphism, these canonical connections are not weakly stable.

Introduction. The subject of *Yang-Mills* solutions has been developed in the last few years by geometrical treatments [1], [3]. In this note, we study a *self-duality* condition and also a *stability* condition on an invariant bundle-connection, which is a Yang-Mills solution, on S^4 or $P_2(\mathbf{C})$.

We show first that the canonical invariant connection on a homogeneous G-vector bundle on a compact symmetric space gives a Yang-Mills solution (Proposition 1). This is available by using the general theory of invariant connections [5].

Secondly, the self-duality is discussed for the canonical invariant connections on the typical 4-spaces S^4 and $P_2(\mathbf{C})$. Its condition is stated as follows: if the isotropy group of the base space is imbedded into the structure group G, then the canonical invariant connection can be represented as the direct sum of a particular self-dual connection and an anti-self-dual one, and on the contrary, it is (anti-) self-dual, when the group is not injectively mapped into G (Theorems 2 and 4). These are easy consequences of the following properties: $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$ and $\mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1)) \cong \mathfrak{su}(2) + \mathbf{R}$.

The third deals with the stability on the canonical invariant connections with respect to the second variational formula of the action integral. We have a nonstability condition (Theorems 3 and 5). Namely, let E be a homogeneous G-vector bundle over S^4 or $P_2(\mathbf{C})$. If E has a nonzero invariant $\mathfrak{g}$-valued 1-form, then the canonical invariant connection is not weakly stable.

We remark that there remains the quesion of whether any Yang-Mills solutions on $SU(2)$- or $SU(3)$-vector bundles over S^4 is (anti-) self-dual [2]. Note that any Yang-Mills solutions on $SU(2)$- or $SU(3)$-vector bundles over S^4 is (anti-) self-dual, if it is assumed weakly stable [3].

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1. Yang-Mills equations. For basic references, see Atiyah et al. [1] and Bourguignon et al. [3].

Let M be a compact oriented Riemannian manifold and P a G-principal bundle over M (i.e., a principal bundle with structure group G), where G is assumed a compact simple Lie group. Let ω be an Ehresmann connection on P with curvature form $\Omega^\omega = d\omega + \frac{1}{2}[\omega, \omega]$. Each connection ω well defines a differential operator $\nabla^\omega = \nabla$; $\Gamma(E) \to \Gamma(\Lambda^1 \otimes E)$ for a G-vector bundle $E = P \times_{(G,\rho)} E^0$ as follows. For any local section s of P, we have a local frame $\{e_i\}$ of E by $e_i = s \cdot e_i^0$, where $\{e_i^0\}$ is a frame of E^0, and a connection matrix $\{\omega_{ij}\}$ by applying the representation ρ to the pull back 1-form $s^*\omega$. Then ∇^ω is well defined by $\nabla^\omega e_i = \sum \omega_{ij} \otimes e_j$. It follows that if s_t is a horizontal lift of a curve x_t in M ($\omega(\dot{s}_t) = 0$), then $s_t \cdot e^0$ is a parallel section of E ($\nabla_{\dot{x}_t}(s_t \cdot e^0) = 0$) for any e^0 in E^0.

The curvature tensor $R^{\nabla^\omega} = R^\omega$ of ∇^ω is the End(E)-valued 2-form defined by $R^\omega(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. R^ω is also defined directly by applying the ρ to the $\mathfrak{g}$-valued 2-form $s^*\Omega^\omega$. From now on, ρ: $G \to \mathrm{GL}(E^0)$ is assumed locally faithful. We can identify R^ω with $s^*\Omega^\omega$. Since $(sg)^*\Omega^\omega = \mathrm{Ad}(g^{-1})(s^*\Omega^\omega)$, $s^*\Omega^\omega$ is considered as a $\mathfrak{g}_P$-valued 2-form, where $\mathfrak{g}_P = P \times_{(G,\mathrm{Ad})} \mathfrak{g}$. Thus, the *action integral* $\int_M |R^\omega|^2\, dv$ is well defined by using the Killing form $(\cdot, \cdot)$ of $\mathfrak{g}$; here

$$|R^\omega|^2 = -\sum_{i<j} \left(s^*\Omega^\omega(e_i, e_j), s^*\Omega^\omega(e_i, e_j)\right),$$

where $\{e_i\}$ is an orthonormal basis of TM.

A critical connection ω of the action integral satisfies the so-called *Yang-Mills equation* $\delta^\omega R^\omega = 0$. The operator δ^ω is a mapping from $\Gamma(\Lambda^{p+1} \otimes \mathrm{End}(E))$ to $\Gamma(\Lambda^p \otimes \mathrm{End}(E))$ defined by $\delta^\omega = -* \circ d^\omega \circ *$; here $*$ is the Hodge star operator given by the orientation of M and d^ω is the exterior covariant differentiation from $\Gamma(\Lambda^p \otimes \mathrm{End}(E))$ to $\Gamma(\Lambda^{p+1} \otimes \mathrm{End}(E))$, defined as $d^\omega(\theta \otimes \Psi) = d\theta \otimes \Psi + (-1)^p \theta \wedge \nabla\Psi$ (Λ^p is the bundle of p-forms). And a connection ω is called a *Yang-Mills solution* iff it satisfies this equation.

We have the following explicit expression of δ^ω [3]:

$$(\delta^\omega\Psi)(X_1, \ldots, X_p) = -\sum_j (\tilde{\nabla}_{e_j}\Psi)(e_j, X_1, \ldots, X_p)$$

for Ψ in $\Gamma(\Lambda^{p+1} \otimes \mathrm{End}(E))$; here $\tilde{\nabla}\Psi$ is defined by

$$(\tilde{\nabla}_Y\Psi)(X_0, \ldots, X_p) = \nabla_Y(\Psi(X_0, \ldots, X_p)) - \sum_k \Psi(X_0, \ldots, D_Y X_k, \ldots, X_p)$$

(D denotes the Riemannian connection of M). From this, we have a trivial remark, that is, if a connection has the parallel curvature ($\tilde{\nabla} R^\omega = 0$), then it gives a Yang-Mills solution.

Suppose now that ω is a Yang-Mills solution. Then, we have the following description due to [3] for the second variational formula of the action integral. For a deformation ω_t of ω with $(d/dt)(\omega_t)_{t=0} = A \in \Gamma(\Lambda^1 \otimes \mathfrak{g}_P)$, $\delta^\omega A = 0$,

$$\frac{1}{2}\frac{d^2}{dt^2}\int_M |R^{(\omega_t)}|^2\, dv_{|t=0} = \int_M (\mathfrak{S}^\omega(A), A)\, dv;$$

here $\mathfrak{S}^\omega: \Gamma(\Lambda^1 \otimes \mathfrak{g}_P) \to \Gamma(\Lambda^1 \otimes \mathfrak{g}_P)$ is defined by

$$\mathfrak{S}^\omega(A) = \delta^\omega d^\omega A + \sum_j \left[R^\omega(e_j, \cdot), A(e_j) \right].$$

A Yang-Mills solution ω is *weakly stable* iff $\int_M (\mathfrak{S}^\omega(A), A)\, dv \geqslant 0$ for any infinitesimal deformation $A \in \Gamma(\Lambda^1 \otimes \mathfrak{g}_P)$, $\delta^\omega A = 0$.

Let M be 4-dimensional. Then, since $*^2 = 1$ on Λ^2, Λ^2 is split into the sum $\Lambda^2 = \Lambda_+ + \Lambda_-$, where Λ_+ (respectively, Λ_-) is the eigenspace of $*$ corresponding to $+1$ (-1). Note that the $*$ commutes with any orientation-preserving transformation of M.

A connection ω is called (*anti-*) *self-dual* iff $* R^\omega = R^\omega$ (resp., $* R^\omega = -R^\omega$). Since $d^\omega R^\omega = 0$ (Bianchi's identity), each (anti-) self-dual connection gives a Yang-Mills solution. And also we have the inequality

$$\int_M |R^\omega|^2 = \int \left(|R_+^\omega|^2 + |R_-^\omega|^2\right) \geqslant \int \left(|R_+^\omega|^2 - |R_-^\omega|^2\right) = 4\pi^2 \operatorname{Pont}_1(E),$$

$R^\omega = R_+^\omega + R_-^\omega$, $R_\pm^\omega \in \Gamma(\Lambda_\pm \otimes \mathfrak{g}_P)$ and the equality occurs iff $R_-^\omega = 0$, that is, ω is self-dual. Thus, the self-dual connections give the absolute minimum for the action integral, hence these are naturally weakly stable.

2. Homogeneous vector bundles and invariant connections. Let $M = K/H$ be a compact oriented Riemannian homogeneous space and P a G-principal bundle over M such that K acts on P as automorphisms. Fix u_o in P over $o = eH$ in M. The K-action induces the *isotropy homomorphism* $\lambda: H \to G$ by $\tilde{h}(u_o) = u_o \cdot \lambda(h)$, where $\tilde{h}$ is an automorphism of P induced by h (see p. 105 in [5]). Let $E = P \times_{(G,\rho)} E^0$ be a vector bundle associated with P through a locally faithful representation $\rho: G \to GL(E^0)$. Then E is homogeneous and isomorphic with $K \times_\tau E^0$, where $\tau = \rho \circ \lambda: H \to GL(E^0)$, and conversely each homomorphism $\lambda: H \to G$ induces a homogeneous G-vector bundle [7].

A connection ω on P is called *invariant* iff $\tilde{k}^*\omega = \omega$ for all k in K. Then, we obtain a one-to-one correspondence between {K-invariant connections ω on P} and

$$\operatorname{Hom}_H(\mathfrak{m}, \mathfrak{g}) = \{\text{linear mappings } \Lambda: \mathfrak{m} \to \mathfrak{g} \text{ such that } \Lambda(\operatorname{Ad}(h)X) = \operatorname{Ad}(\lambda(h))\Lambda(X),\ X \in \mathfrak{m} \text{ and } h \in H\},$$

and the correspondence is given by

$$\omega_{u_0}(\tilde{X}) = \lambda(X_\mathfrak{h}) + \Lambda(X_\mathfrak{m}) \quad \text{for } X = X_\mathfrak{h} + X_\mathfrak{m} \in \mathfrak{k} = \mathfrak{h} + \mathfrak{m},$$

where $\tilde{X}$ is the vector field in P induced by X and the curvature form Ω^ω of ω has the following expression:

$$2\Omega^\omega_{u_0}(\tilde{X}, \tilde{Y}) = \left[\lambda(X_\mathfrak{h}) + \Lambda(X_\mathfrak{m}), \lambda(Y_\mathfrak{h}) + \Lambda(Y_\mathfrak{m})\right] - \lambda\left([X, Y]_\mathfrak{h}\right) - \Lambda\left([X, Y]_\mathfrak{m}\right) \quad \text{for } X \text{ and } Y \text{ in } \mathfrak{k}$$

(see Theorem 11.7 in [5]). Here $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ is a reductive decomposition of $\mathfrak{k}$. Each bundle P always admits a particular K-invariant connection corresponding to $\Lambda = 0$, which is called *canonical*. Its curvature satisfies $2\Omega^\omega_{u_0}(\tilde{X}, \tilde{Y}) = -\lambda([X, Y]_\mathfrak{h})$ for X and Y in $\mathfrak{m}$.

PROPOSITION 1. *Let $M = K/H$ be a compact oriented Riemannian symmetric space and E a homogeneous G-vector bundle associated with a K-invariant G-principal bundle P. Then the canonical invariant connection has parallel curvature, and hence it gives a Yang-Mills solution.*

PROOF. Let $f_t = \exp tX$ be the 1-parameter subgroup of K generated by $X \in \mathfrak{m}$ and $\tilde{f}_t$ the 1-parameter group of transformations of P induced by f_t. The tangent vector of the orbit $\tilde{f}_t(u_o)$ in P is $\tilde{X}$ at $f_t(u_o)$. Because the connection ω is canonical, $\omega_{u_0}(\tilde{X}) = 0$, that is, the orbit is horizontal. Hence, the section $\tilde{f}_t(u_o) \cdot e^0$ of E ($e^0 \in E^0$) is parallel along $x_t = f_t(o)$. Since M is symmetric, the Riemannian connection is also canonical (Theorem 3.1 in [6]). Then, $f_t(a_o) \cdot a_o^{-1}Y$ is parallel to Y in TM_o along x_t for any frame a_o at o. Therefore, we have at $t = 0$,

$$\begin{aligned}(\tilde{\nabla}_{\dot{x}_t}R^{\omega})(\tilde{X}_1, \tilde{X}_2)\tilde{\psi} &= \nabla_{\dot{x}_t}\big(R^{\omega}(\tilde{X}_1, \tilde{X}_2)\tilde{\psi}\big) - R^{\omega}\big(D_{\dot{x}_t}\tilde{X}_1, \tilde{X}_2\big)\tilde{\psi}\\ &\quad - R^{\omega}\big(\tilde{X}_1, D_{\dot{x}_t}\tilde{X}_2\big)\tilde{\psi} - R^{\omega}(\tilde{X}_1, \tilde{X}_2)\nabla_{\dot{x}_t}\tilde{\psi}\\ &= \nabla_{\dot{x}_t}\big(R^{\omega}(\tilde{X}_1, \tilde{X}_2)\tilde{\psi}\big)\end{aligned}$$

with respect to the parallel extensions $\tilde{X}_i(t) = f_t(a_o) \cdot a_o^{-1}X_i$, $i = 1, 2$, and $\tilde{\psi}(t) = \tilde{f}_t(u_o) \cdot u_o^{-1}\psi$. Since R^{ω} is invariant,

$$\nabla_{x_t}\big(R^{\omega}(\tilde{X}_1, \tilde{X}_2)\tilde{\psi}\big) = \nabla_{x_t}\big(\tilde{f}_t(u_o) \cdot u_o^{-1}(R^{\omega}(X_1, X_2)_j)\big) = 0;$$

hence R^{ω} is parallel.

Note. For any $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$, the invariant $\mathfrak{g}_P$-valued 1-form A induced by Λ is shown to be parallel by an argument similar to that in the proof.

3. Self-duality condition. Now consider, in this section, invariant connections on the 4-sphere $S^4 = SO(5)/SO(4)$. The algebra $\mathfrak{so}(5)$ has the reductive decomposition: $\mathfrak{so}(5) = \mathfrak{h} + \mathfrak{m}$, $\mathfrak{h} = \mathfrak{so}(4)$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. And in this case, we have the following decomposition of $\mathfrak{h}$. $\mathfrak{h} = \mathfrak{su}(2)^{(1)} + \mathfrak{su}(2)^{(2)}$, where $\mathfrak{su}(2)^{(i)}$ is the subalgebra spanned by $\{A^i, B^i, C^i\}$: $A^i = E_{2,3} + (-1)^{i+1}E_{4,5}$, $B^i = E_{3,4} + (-1)^{i+1}E_{2,5}$ and $C^i = E_{2,4} + (-1)^iE_{3,5}$, $i = 1, 2$ ($E_{i,j}$ denotes the matrx in $\mathfrak{so}(5)$ whose entries satisfy $(E_{i,j})_{k,l} = 0$ if $\{k, l\} \neq \{i, j\}$ and $(E_{i,j})_{i,j} = 1$ and $(E_{i,j})_{j,i} = -1$ for $i < j$). Both $\mathfrak{su}(2)^{(i)}$ are isomorphic with $\mathfrak{su}(2)$. The subspace $\mathfrak{m}$ is spanned by $X_i = E_{1,i+1}$, $i = 1, \ldots, 4$. The orientation of S^4 is fixed once and for all by this frame.

Let E be a homogeneous G-vector bundle over S^4 (G is a compact simple Lie group). The $SO(5)$-action on E induces the isotropy homomorphism $\lambda\colon \mathfrak{h} \to \mathfrak{g}$.

The following gives a self-duality condition for the canonical invariant connections.

THEOREM 2. (i) *The canonical invariant connection is (anti-) self-dual iff the λ vanishes on the second (first) factor of $\mathfrak{h}$.*

(ii) *If $\mathfrak{h}$ is imbedded into $\mathfrak{g}$ by the λ, then the canonical connection is not (anti-) self-dual, and hence,*

(iii) *If $\mathfrak{g}$ is either $\mathfrak{so}(5)$, the algebra of G_2 or of rank $r(G) \geqslant 3$, there are homogeneous G-vector bundles whose canonical connections are not (anti-) self-dual, and on the contrary, if $\mathfrak{g}$ is $\mathfrak{su}(2)$ or $\mathfrak{su}(3)$, the canonical connection is (anti-) self-dual.*

PROOF. (i) The curvature R^ω of the canonical connection ω is written as $R^\omega(X, Y) = -\frac{1}{2}\lambda([X, Y])$, $X, Y \in \mathfrak{m}$. The condition that R^ω is (anti-) self-dual is the following: $R^\omega(X_1, X_2) = \pm R^\omega(X_3, X_4)$, $R^\omega(X_1, X_3) = \mp R^\omega(X_2, X_4)$ and $R^\omega(X_1, X_4) = \pm R^\omega(X_2, X_3)$. By the bracket computation, we have

$$[X_1, X_2] = -\tfrac{1}{2}(A^1 + A^2), \quad [X_3, X_4] = -\tfrac{1}{2}(A^1 - A^2),$$
$$[X_1, X_3] = -\tfrac{1}{2}(C^1 + C^2), \quad [X_4, X_2] = -\tfrac{1}{2}(C^1 - C^2),$$
$$[X_1, X_4] = -\tfrac{1}{2}(B^1 - B^2), \quad [X_2, X_3] = -\tfrac{1}{2}(B^1 + B^2),$$

which implies that the (anti-) self-duality of R^ω is equivalent to $\lambda(A^2) = \lambda(B^2) = \lambda(C^2) = 0$ ($\lambda(A^1) = \lambda(B^1) = \lambda(C^1) = 0$). (ii) is also obtained by this argument.

(iii) Assume that the rank $r(G) \geqslant 3$. As is well known, there are simple roots α_i and α_j such that $(\alpha_i, \alpha_j) = 0$, $i < j$. The root vectors corresponding to α_i (resp. α_j) generate the subalgebra $\mathfrak{h}_i$ (resp. $\mathfrak{h}_j$) of $\mathfrak{g}$, which is isomorphic to $\mathfrak{su}(2)$. Since $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$, we have the injective homomorphism $\lambda\colon \mathfrak{h} \to \mathfrak{g}$ such that $\mathfrak{su}(2)^{(1)}$ and $\mathfrak{su}(2)^{(2)}$ are mapped onto $\mathfrak{h}_i$ and $\mathfrak{h}_j$, respectively. Thus, by (ii), the homogeneous G-vector bundle over S^4 induced by the λ has the non- (anti-) self-dual canonical connection. If $\mathfrak{g}$ is $\mathfrak{so}(5)$ or the algebra of G_2, $\mathfrak{h} = \mathfrak{so}(4)$ is canonically imbedded into $\mathfrak{g}$. Hence by (ii), we have the same conclusion. Moreover, if $\mathfrak{g}$ is $\mathfrak{su}(2)$ or $\mathfrak{su}(3)$, any homomorphism from $\mathfrak{h}$ to $\mathfrak{g}$ is not imbedded. Thus, the last part of (iii) is verified by (i).

REMARKS. (i) The holonomy group of the G-canonical connection is generated by the image of $\mathfrak{su}(2) + \mathfrak{su}(2)$ through λ (see Theorem 11.8 in **[5]**).

(ii) If $SO(4)$ is imbedded into a simple G, then $\mathrm{Pont}_1(E) = 0$ for any homogeneous G-vector bundle E constructed by this imbedding. This fact is as follows. The curvature of the canonical connection is the image of the curvature of the standard Riemannian connection on TS^4. Thus, $\mathrm{Pont}_1(E)$ is a scalar multiple of $\mathrm{Pont}_1(S^4) = 0$.

4. Weak stability. In this section we discuss the weak stability for Yang-Mills solutions given by the canonical invariant connections which are not (anti-) self-dual.

We fix a Λ in $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$. Since Λ commutes with the action of H, the $\mathfrak{g}_P$-valued 1-form A induced by Λ is parallel, by the Note in §2, hence $\delta^\omega A = d^\omega A = 0$. Then $\omega_t = \omega + tA$ gives a deformation of ω. Since $R^{(\omega_t)}$ is invariant under K, $|R^{(\omega_t)}|^2$ is constant. Thus, we have the following:

$$\frac{1}{2}\,\frac{d^2}{dt^2}\int_{S^4}|R^{(\omega_t)}|^2\,dv_{|t=0} = \frac{1}{2}\,\mathrm{vol}(S^4)\frac{d^2}{dt^2}\left\{|R^{(\omega_t)}|^2 \text{ at the origin } o\right\}_{|t=0}$$
$$= \mathrm{vol}(S^4)\left(R^\omega, [\Lambda, \Lambda]\right).$$

Here we used the formula

$$R^{(\omega_t)}(X, Y) = R^\omega(X, Y) + (t^2/2)[\Lambda(X), \Lambda(Y)]$$

at o.

THEOREM 3. *Let E be a homogeneous G-vector bundle over S^4 induced by an injective homomorphism λ of H to G. Then the canonical connection ω satisfies $\int_{S^4}(\mathfrak{S}^\omega A, A)\, dv < 0$ for any invariant $\mathfrak{g}_P$-valued 1-form A induced by Λ in* $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$, $\Lambda \neq 0$. *Therefore, if* $\dim \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g}) \geqslant 1$, *then ω is not weakly stable.*

PROOF. By the definition of $\mathfrak{S}^\omega$, we have

$$\int_{S^4}(\mathfrak{S}^\omega A, A)\, dv = \frac{1}{2}\frac{d^2}{dt^2}\int_{S^4}|R^{(\omega_t)}|^2\, dv_{|t=0}$$

for a deformation ω_t with $(d/dt)(\omega_t)_{|t=0} = A$. Since $R^\omega(X, Y) = -\frac{1}{2}\lambda[X, Y]$ at o and $\Lambda[Z, X] = [\lambda Z, \Lambda X]$ for $Z \in \mathfrak{h}$ and $X \in \mathfrak{m}$,

$$\begin{aligned}(R^\omega, [\Lambda, \Lambda]) &= \sum_{i<j}(R^\omega(e_i, e_j), [\Lambda e_i, \Lambda e_j])\\ &= -\frac{1}{2}\sum_{i<j}(\lambda[e_i, e_j], [\Lambda e_i, \Lambda e_j])\\ &= \frac{1}{2}\sum_{i<j}([\Lambda e_i, \lambda[e_i, e_j]], \Lambda e_j)\\ &= \frac{1}{4}\sum_{i,j}(\Lambda[e_i, [e_i, e_j]], \Lambda e_j)\\ &= -\frac{3}{4}\sum_{i=1}^{4}|\Lambda(e_i)|^2;\end{aligned}$$

here $\{e_i\}$, $i = 1, \ldots, 4$, is the orthonormal basis of $\mathfrak{m}$. Thus, if $\Lambda \neq 0$, then $\int_{S^4}(\mathfrak{S}^\omega A, A)\, dv < 0$.

REMARKS. (i) Let μ: $SO(5) \to G$ (or, more precisely, $\mathrm{Spin}(5) \to G$) be an injective homomorphism. Then, $\lambda = \mu_{|SO(4)}$ induces a homogeneous G-vector bundle over S^4. This bundle E is the image of the tangent bundle TS^4 and admits the nontrivial $\Lambda = \mu_{|\mathfrak{m}}$ in $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$. Then the canonical connection ω on E is just the image of the standard Riemannian connection ω^0 on S^4 and ω is shown to be not weakly stable by the above theorem, whereas ω^0 minimizes the action integral

$$k \cdot \mathrm{Euler}(TS^4) = \int_{S^4}|R^{\omega^0}|^2\, dv \leqslant \int_{S^4}|R^\omega|^2\, dv$$

for any connections ω [4]. If the structure group is enlarged from $SO(4)$ to general G, the obstruction induced by the Euler characteristic is removed in contrast to Pont_1, in fact the integral can be decreased by the theorem.

(ii) It is well known that for any compact simple Lie group G of rank $\geqslant 3$, there is an injective homomorphism of Spin(5) into G.

5. The case of $P_2(\mathbf{C})$. We can also discuss the similar argument for another 4-dim symmetric space of compact type; the 2-dim complex projective space $P_2(\mathbf{C}) = SU(3)/S(U(2) \times U(1))$. The following is easily shown.

THEOREM 4. *Let λ be a homomorphism of $S(U(2) \times U(1))$ to G. Then*:

(i) *the canonical connection on a homogeneous G-vector bundle induced by the λ is self-dual iff $\lambda_{|\mathfrak{su}(2)} = 0$ and is anti-self-dual iff $\lambda_{|\mathbf{R}} = 0$; and*

(ii) *when $r(G) \geqslant 2$, there is a homogeneous G-vector bundle whose canonical connection is not (anti-) self-dual.*

$S(U(2) \times U(1))$ has the Lie algebra $\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1))$, spanned by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix};$$

hence it is isomorphic to $\mathfrak{su}(2) + \mathbf{R}$. The orientation is the standard one given by $\{X_i\}$, $1 \leqslant i \leqslant 4$, where

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and

$$X_4 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}.$$

PROOF. Since $[X_1, X_2] = D - C$, $[X_3, X_4] = D + C$, $[X_1, X_3] = [X_2, X_4] = -A$ and $[X_2, X_3] = -[X_1, X_4] = B$, (i) is obtained.

(ii) is evident from the fact that there is an injective homomorphism of $S(U(2) \times U(1))$ to any simple G of rank $\geqslant 2$.

THEOREM 5. *Let E be a homogeneous G-vector bundle over $P_2(\mathbf{C})$ induced by an injective homomorphism λ of $S(U(2) \times U(1))$ to G. Then the canonical connection ω satisfies $\int_{P_2(\mathbf{C})}(\mathfrak{S}^\omega A, A)\, dv < 0$ for any invariant $\mathfrak{g}_P$-valued 1-form A induced by Λ in* $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$, $\Lambda \neq 0$. *Thus, if* $\dim \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g}) \geqslant 1$, *$\omega$ is not weakly stable.*

PROOF. The proof is given in the same manner as the proof of Theorem 3. After a simple bracket calculation, we have

$$-\sum_{i<j} (\lambda[X_i, X_j], [\Lambda X_i, \Lambda X_j])$$
$$= -\{3|\Lambda(X_1)|^2 + 3|\Lambda(X_2)|^2 + 2|\Lambda(X_3)|^2 + 3|\Lambda(X_4)|^2\} < 0$$

for nonzero Λ in $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$. Hence $\int_{P_2(\mathbf{C})}(\mathfrak{S}^\omega A, A)\, dv < 0$.

REMARKS. (i) It may follow from the fact that $S(U(2) \times U(1))$ contains the nontrivial normal subgroup $U(1)$ that the coefficient of $|\Lambda(X_3)|^2$ differs from the other coefficients.

(ii) $SU(3)$ can be imbedded into G of rank $\geqslant 3$. Thus, this imbedding restricted to $S(U(2) \times U(1))$ induces non- (anti-) self-dual canonical connection, which is actually not weakly stable, from the above theorems.

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ADDED IN PROOF. The following fact is shown in [4]. Let M be an orientable Riemannian homogeneous 4-space. If ω is a weakly stable Yang-Mills $SU(2)$-connection on M, then it is self-dual or anti-self-dual.

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